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A note on inequalities for the masses of the lightest $\pi\pi$ resonances in multicoloured QCD

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Abstract. We derive and analyse inequalities relating masses of the lightest $\pi\pi$ resonances (ρ and σ) to coupling constants of the effective chiral Lagrangian in the limit of a large number of colours.

PACS. 11.15.Pg Expansions for large numbers of components (*e.g.*, $1/N_c$ expansions) – 14.40.-n Mesons

1. – The issue of the existence of the light scalar resonance (we call it σ in what follows) is one of the most controversial questions in the meson spectroscopy (for a review of the scalar meson spectroscopy see the note on scalar mesons by S. Spanier and N. Tornqvist in *Review of Particle Physics* [1][1]). Recently, this topic attracted a considerable interest, see refs. [3–40].

In these notes we analyse the sum rules relating low-energy constants (LECs) of the effective chiral Lagrangian (EChL) to the resonance spectrum parameters in the limit of a large number of colours [41,42][2].

We shall show that from these sum rules one can derive a set of inequalities, *e.g.*, such as:

$$M_\sigma^2(3L_2+L_3)+M_\rho^2 L_2 \le \frac{F_\pi^2}{4}\,, \tag{1}$$

where L_i are the coupling constants of the fourth-order EChL [45], M_ρ is the mass of the lightest isovector resonance (ρ-meson), M_σ is the mass of the lightest isoscalar resonance (σ-meson), and $F_\pi \approx 93$ MeV is the pion decay constant. This inequality, apart from applications for estimates of the σ-meson mass from above, demonstrates that properties of the resonance spectrum are in close relations with properties of chiral symmetry breaking. Additionally inequalities of the type (1), due to their large-N_c nature, can give us a possibility to study systematically the controversial nature of the σ-meson. Below we give derivation of the inequality (1) as well as its enhancements.

2. – Following the ideas of refs. [41,42] one can easily derive the following set of large-N_c sum rules, relating the constants of the effective chiral Lagrangian L_i[3] to the parameters of the resonance spectrum

$$\begin{aligned}
1+O(m_\pi^4) &= \sum \frac{F_0^2 V_0}{\left[M_0^2-2m_\pi^2\right]^2}+\sum \frac{F_0^2 V_1}{\left[M_1^2-2m_\pi^2\right]^2}\,,\\
3L_2+L_3+\alpha m_\pi^2+O(m_\pi^4) &= \frac{F_0^4}{4}\sum \frac{V_0}{\left[M_0^2-2m_\pi^2\right]^3}\,,\\
L_2+\beta m_\pi^2+O(m_\pi^4) &= \frac{F_0^4}{4}\sum \frac{V_1}{\left[M_1^2-2m_\pi^2\right]^3}\,.
\end{aligned} \tag{2}$$

Here M_I are the masses of pion-pion resonances with isospin I, and V_I the corresponding residues. The latter are related to the $\pi\pi$ resonance width $\Gamma(R\to\pi\pi)$ via

$$V_0=\frac{2}{3}16\pi(2J+1)\frac{M_0^2}{\sqrt{M_0^2-4m_\pi^2}}\Gamma(R\to\pi\pi)\,, \tag{3}$$

$$V_1=16\pi(2J+1)\frac{M_1^2}{\sqrt{M_1^2-4m_\pi^2}}\Gamma(R\to\pi\pi)\,, \tag{4}$$

where J is the resonance spin. The constant $F_0\approx 88$ MeV is the pion decay constant in the chiral limit. The constants α,β are related to low-energy coefficients (LECs) of the sixth-order EChL. We use estimates for the LECs of the sixth-order EChL obtained in refs. [46,47] from the chiral expansion of the dual (string) models

$$\alpha m_\pi^2 \approx 0.18\cdot 10^{-3}, \qquad \beta m_\pi^2 \approx 0.05\cdot 10^{-3}\,. \tag{5}$$

In refs. [41,42] the sum rules (2) have been derived in the chiral limit ($m_\pi=0$). Here for completeness we give

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[1] Related problems are also discussed in the review article [2].
[2] In [43] it is shown that those sum rules can be derived from the general postulates of the effective theory without referring to large-N_c limit; see also [44].
[3] Note that LECs L_i are scale independent in the large-N_c limit.

short derivation of the first sum rule in eqs. (2). Writing down the dispersion relation for $\pi\pi$ amplitude at fixed t one can easily show that the low-energy constants satisfy the following dispersion sum rules:

$$\frac{1}{F_0^2}+\frac{8m_\pi^2}{F_0^4}(\lambda-2)(2L_2+L_3)+O(m_\pi^4)+O\left(\frac{1}{N_c^2}\right)= \\ \frac{1}{\pi}\int_{4m_\pi^2}^\infty \mathrm{d}x\Bigg\{\frac{1}{3[x-\lambda m_\pi^2]^2}(\rho^{(0)}-\rho^{(2)}) \\ +\frac{1}{2[x-(4-\lambda)m_\pi^2]^2}(\rho^{(1)}-\rho^{(2)})\Bigg\}, \tag{6}$$

where $\rho^{(I)}=32\pi\sum_{l=0}^\infty(2l+1)\mathrm{Im}\ t_l^I(s+i0)$ and

$$t_l^I(s+i0)=\sqrt{\frac{s}{s-4m_\pi^2}}\frac{1}{2i}[e^{2i\delta_l^I(s)}-1] \tag{7}$$

is a partial-wave $\pi\pi$ amplitude with isospin I in s-channel, $\delta_l^I(s)$ is the corresponding phase shift. In deriving dispersive sum rules, eq. (6), we used Regge asympotic conditions imposed on the $\pi\pi$ scattering amplitudes with fixed isospin in the t-channel

$$T_t^0(s,t)\sim s^{\alpha_0(t)}, \qquad \alpha_0(0)\approx 1, \\ T_t^1(s,t)\sim s^{\alpha_1(t)}, \qquad \alpha_1(0)\approx \frac{1}{2}, \\ T_t^2(s,t)\sim s^{\alpha_2(t)}, \qquad \alpha_2(0)<0,$$

and Bose-symmetry requirements.

The spectral densities $\rho^{(I)}$ in the large-N_c limit have the form

$$\rho^{(0)}=3\pi\sum_{I=0\ \mathrm{res.}}V_0\delta(s-M_0^2), \tag{8}$$

$$\rho^{(1)}=2\pi\sum_{I=1\ \mathrm{res.}}V_1\delta(s-M_1^2), \tag{9}$$

$$\rho^{(2)}=0 \qquad \text{(absence of isospin-two resonances!)}. \tag{10}$$

Substituting these large-N_c spectral densities into dispersive sum rule (6) and choosing the parameter $\lambda=2$, we obtain the first sum rule of eqs. (2)

3. – From the sum rules, eqs. (2), one can immediately obtain the following obvious inequalities:

$$3L_3+L_2+\alpha m_\pi^2>0, \\ L_2+\beta m_\pi^2>\frac{V_\rho F_0^4}{4(M_\rho^2-2m_\pi^2)^3}\approx 1.66\cdot 10^{-3}, \tag{11}$$

where M_ρ and V_ρ stand for the mass and residue of the lightest isovector $\pi\pi$ resonance (ρ-meson). Further, noting that

$$\sum_{I=0}\frac{V_0}{[M_0^2-2m_\pi^2]^k}>\left[M_\sigma^2-2m_\pi^2\right]\sum_{I=0}\frac{V_0}{[M_0^2-2m_\pi^2]^{k+1}}, \tag{12}$$

where $k\geq 2$ and M_σ is a mass of the lightest isoscalar (scalar) resonance, we obtain the following inequality:

$$M_\sigma^2\left(3L_3+L_2+\alpha m_\pi^2\right)+M_\rho^2\left(L_2+\beta m_\pi^2\right)< \\ \frac{F_0^2}{4}+2m_\pi^2\left(4L_2+L_3\right). \tag{13}$$

This inequality provides us with a nice example of nontrivial relations between the parameters of the resonance spectrum and low-energy constants of EChL. The model-independent large-N_c inequality (13) can be used for the estimates of the σ-meson mass from above (see below), as well as for consistency checks of various models of low-energy QCD in the large-N_c limit.

Parameters of the EChL in the large-N_c limit have been calculated in various models of the low-energy QCD [48–53]. We shall use parameters from the analysis of the EChL coupling constants in the large-N_c limit done in [53] (the error bars take into account different values of the constants obtained in the fits performed in [53]):

$$L_2= (1.6\pm 0.1)\cdot 10^{-3}, \\ L_3=-(4\pm 1)\cdot 10^{-3}, \tag{14}$$

These values are close to those obtained from the phenomenological analysis [45,54], which shows that the $1/N_c$ corrections to low-energy coefficients L_i are rather small.

Due to the inequality, eq. (11), the value of L_2 cannot be below $1.63\cdot 10^{-3}$, therefore we shall use this minimal value of $L_2=1.63\cdot 10^{-3}$ lying in the range given by eq. (14)[4]. The error of calculation of L_3 is bigger. Also, the errors of L_2 and L_3 are strongly correlated. In order to make an estimation of the M_σ based on inequality, eqs. (13), we use first the relation $2L_2+L_3=0$ which follows from integration of the non-topological chiral anomaly [48–50] and from the low-energy limit of the dual-resonance (string) models [46]. Using the above values of L_2 and L_3, we obtain from eqs. (13)

$$M_\sigma<770\ \mathrm{MeV}, \qquad \text{if } 2L_2+L_3=0. \tag{15}$$

This is the upper bound for the lightest isoscalar resonance, if one assumes the relation $2L_2+L_3=0$. To consider the more general case, we derive the upper limit on M_σ as a function of the parameter Δ defined as follows:

$$\Delta=-\frac{2L_2+L_3}{L_2}. \tag{16}$$

The value of this parameter is zero for EChL obtained by integration of non-topological chiral anomaly [48–51] as well as for EChL obtained by chiral expansion of the dual-resonance (string) models [46]. In the large-N_c–based model of ref. [52] the value of Δ is fixed in terms of gluon condensate and constituent quark mass $m_Q\approx 0.35$ GeV as $\Delta=\frac{\pi^2\langle\frac{\alpha_s}{\pi}G^2\rangle}{5N_c m_Q^4}\approx 0.3$. The value of LECs obtained in ref. [55] corresponds to $\Delta=5/8=0.625$. In any case

[4] Note that for larger values of L_2 the bounds on M_σ discussed below are stronger.

the value of Δ cannot exceed unity due to the inequality (11). Experimentally, the parameter Δ is constrained by the ratio of the D-wave pion scattering lengths

$$\Delta = -3\ \frac{a_2^2}{a_2^0} + O(m_\pi^2) \approx -0.2 \pm 0.6\,, \tag{17}$$

where we took the experimental values of the D-wave scattering lengths from ref. [56].

Now it is easy to derive from the inequality (13) the upper bound for the σ-meson mass as a function of the parameter Δ. This function at small values of Δ takes the form

$$M_\sigma < 770\left[1 + 0.42\Delta + 0.29\Delta^2 + O(\Delta^3)\right]\ \text{MeV}\,. \tag{18}$$

We see that the upper bound for the σ-meson mass is sensitive to the sign of the parameter Δ (see definition (16)). Therefore the values of LECs of the fourth-order EChL can give us a valuable information about the lightest scalar meson in the spectrum of QCD.

4. – In the case when one possesses an additional information (masses and widths of resonances) on the excited meson spectrum (mesons heavier than σ in the isoscalar channel and ρ in isovector one) the inequality (13) can be enhanced. Let us call the excited resonances for which we have the additional information about their masses and widths as *known*. With this additional information the inequality (13) can be enhanced as follows:

$$\begin{aligned}
&M_\sigma^2\left(3L_3 + L_2 + \alpha m_\pi^2 - \sum_{\text{known}} \frac{F_0^4 V_0}{4\left[M_0^2 - 2m_\pi^2\right]^3}\right) \\
&+M_\rho^2\left(L_2 + \beta m_\pi^2 - \sum_{\text{known}} \frac{F_0^4 V_1}{4\left[M_1^2 - 2m_\pi^2\right]^3}\right) < \\
&\frac{F_0^2}{4} - \sum_{\text{known}} \frac{F_0^4 V_0}{4\left[M_0^2 - 2m_\pi^2\right]^2} - \sum_{\text{known}} \frac{F_0^4 V_1}{4\left[M_1^2 - 2m_\pi^2\right]^2} \\
&+2m_\pi^2\left(4L_2 + L_3 - \sum_{\text{known}} \frac{F_0^4 V_0}{2\left[M_0^2 - 2m_\pi^2\right]^3}\right. \\
&\left. - \sum_{\text{known}} \frac{F_0^4 V_1}{2\left[M_1^2 - 2m_\pi^2\right]^3}\right).
\end{aligned} \tag{19}$$

For the numerical estimates we take, as the *known* resonances, $f_2(1275)$ in the isoscalar channel and $\rho_3(1690)$ in the isovector channel. We do not include other scalar and vector mesons as their nature is not well established and it is not clear whether their dynamics is "leading" in the large-N_c limit. Taking the masses and $\pi\pi$ widths of $f_2(1275)$ and $\rho_3(1690)$ from [1], we obtain the enhancement of the inequality (18)

$$M_\sigma < 665\left[1 + 0.44\Delta + 0.33\Delta^2 + O(\Delta^3)\right]\ \text{MeV}\,. \tag{20}$$

Obviously the inclusion of other resonances, *e.g.* $f_0(980)$, $f_0(1370)$, $f_0(1500)$, ρ', f_4, etc. would lead to lower bound on the mass of σ-meson.

5. – To summarize, we derive the inequalities for the masses of the lightest $\pi\pi$ resonances in the limit of a large number of colours ($N_\text{c} \to \infty$), see eqs. (13),(19). These inequalities put an upper bound on the mass of σ-meson in terms of pion decay constant F_π and the low-energy constants of effective chiral Lagrangian L_2 and L_3. Analysis of these inequlities favours the presence of the light (mass < 750 MeV) scalar state in the meson spectrum of the multicolour QCD.

As a final remark, we note that the sum rules (2) are derived in the limit of a large number of colours, this implies that the exotic mesons (glueballs, four-quark states) do not contribute to the sum rules because their contributions are suppressed by powers of $1/N_\text{c}$. This observation shows that the sum rules (2) can be used for identification of the nature of low-lying scalar mesons. For example, the sum rules in eq. (2) tell us that the leading large-N_c part of the width (read the width of the $q\bar{q}$ and hybrid part) and the mass of the σ-meson should satisfy the following constraint:

$$\frac{32\pi F_0^2 M_\sigma^2 \Gamma\left(\sigma \to \pi\pi\right)}{3\sqrt{M_\sigma^2 - 4m_\pi^2}\left[M_\sigma^2 - 2m_\pi^2\right]^2} \le 1 - \frac{4\left(M_\rho^2 - 2m_\pi^2\right)}{F_0^2} L_2\,. \tag{21}$$

Obviously, other sum rules in eq. (2) and an additional information about resonance spectrum would provide more sophisticated constrains on the parameters of $q\bar{q}$ component of the σ-meson. We shall analyse them elsewhere.

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